MAC-CPTM Situations Project

Situation 62: Absolute Value in the Complex Plane

(Combined CAS Situation 2 and Situation PN3)

CAS Situation 2: Absolute Value and Square Roots
Prepared at Pennsylvania State University
Mid-Atlantic Center for Mathematics Teaching and Learning
19 September 2006 -- Heather Godine, Shiv Karunakaran, and Evan McClintock

Situation PN3: Absolute Value in the Complex Plane
Prepared at University of Georgia
Independent Project
29 July 2006 – Pawel Nazarewicz

Revised at
University of Georgia
Center for Proficiency in Teaching Mathematics
17 November 2008 – Erik Jacobson and Kelly Edenfield

Prompt

A student was working on the task of producing a function that had certain given characteristics. One of those characteristics was that the function should be undefined for values less than 5. Another characteristic was that the range of the function should contain only non-negative values. In the process, the student defined $f(x) = \left|\sqrt{x-5}\right|$ and then evaluated $f(-10)$ using his CAS calculator. The result was 3.872983346. He looked at the calculator screen and whispered, "How can that be?"

Commentary

Two major mathematical ideas arise from this prompt. One is the geometric representation of the complex numbers on a plane. The second is related: the meaning of absolute value in this complex plane. Mathematical Foci 1 and 2 address these ideas. Mathematical Focus 3 discusses linear absolute value equations in the complex plane. Finally, drawing together the major ideas of the Situation, Mathematical Focus 4 verifies that $f(-10) \approx 3.873$ by considering the student-generated function $f(x) = \left|\sqrt{x-5}\right|$ as a composition of three functions.

The Post-Commentary discusses the ambiguous nature of the task in light of the use of a calculator, including a Computer Algebra System (CAS), that can function in complex mode.

Mathematical Foci

Mathematical Focus 1

Complex numbers can be represented as points on the complex plane.

Representing real numbers requires only a one-dimensional system, as we can represent all of the real numbers on a single line. In this way, a real number, x, can be represented by a unique point on the real number line. On the other hand, representing complex numbers requires a plane, where the real numbers are contained on one line and the imaginary numbers are contained on another line perpendicular to the real line. In this way, a complex number $z = x + yi$ can be represented uniquely by a point having coordinates (x,y) on the complex plane. For example, the complex number 2 + 3.5*i* is represented by the point (2,3.5) on the complex plane (Figure 1).

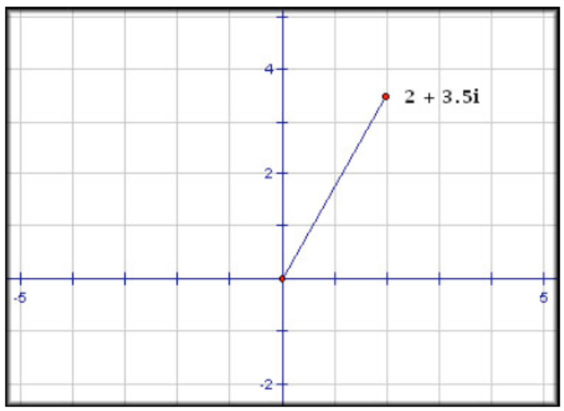

Figure 1

By definition, complex numbers are of the form $z = x + yi$, where x and y are real numbers. Therefore, z is a real number if and only if $y = 0$. In this way, the real numbers are a complete proper subset of the complex numbers.

Mathematical Focus 2

The absolute value of a complex number, $z = x + yi$, is the number's distance from the origin. This distance is called the modulus and is computed by $|z| = \sqrt{x^2 + y^2}$.

As observed before, complex numbers $z = x + yi$, where x and y are real numbers, can be represented by the point (x,y) on the complex plane. The absolute value of a number, including the complex numbers is the defined as the distance the number is from zero. In the complex plane, "zero" is the origin. Therefore, the geometric interpretation of $|z|$ is the distance between the point (x,y) and the origin.

Consider a real number expressed in complex form. This means that $y = 0$ and the complex number can be expressed as $z = x + 0i = x$. Plotting this complex number in the complex plane as $(x, 0)$, one can see that the absolute value of *z*, the distance *z* is from the origin, is *x*.

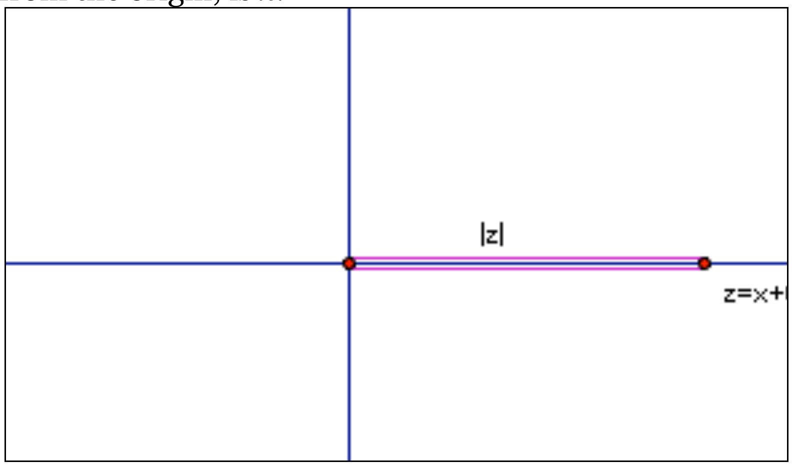

Figure 2

However, consider the general complex number, $z = x + yi$. The distance of *z* from the origin can be determined by dropping a perpendicular from the point *(x, y)* in the complex plane to the real axis. This creates a right triangle with side lengths *x* and *y* and hypotenuse *z*. To determine the length of *z*, the distance the complex number is from the origin, we can apply the Pythagorean Theorem. This yields $|z| = \sqrt{x^2 + y^2}$.

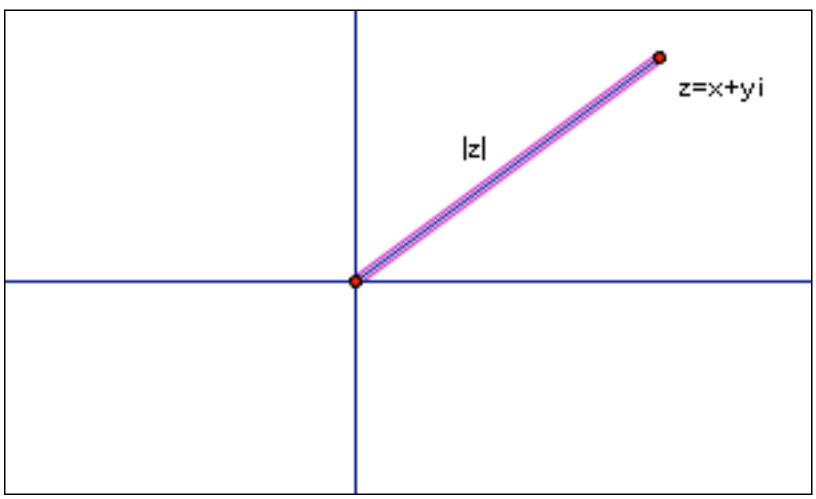

Figure 3

Mathematical Focus 3

In the complex plane, there are infinitely many solutions to linear absolute value equations and these solutions form a circle..

To address the question in the prompt, one must understand that many numbers can have the same absolute value in the complex plane. Consider $|x| = 3$. This means that *x* is 3 units away from 0. In the real numbers, there are two solutions, namely *x* = 3 and *x* = -3. In the complex plane, there are infinitely many solutions for *x*. Each point on the circle of radius 3 centered at the origin is, by definition, 3 units away from point (0,0), that is, 3 units away from 0 + 0i, which is the complex number 0.

We can also solve $|x| = 3$ in the following way:

$$|x| = |a + bi| = \sqrt{a^2 + b^2} = 3$$
$$\Rightarrow a^2 + b^2 = 9$$

Again, we have a circle of radius 3 centered at the origin (Figure 4). Notice that the real line (the horizontal axis) intersects the solution circle twice, once at (-3,0) and again at (3,0). These points correspond to the real numbers -3 and 3.

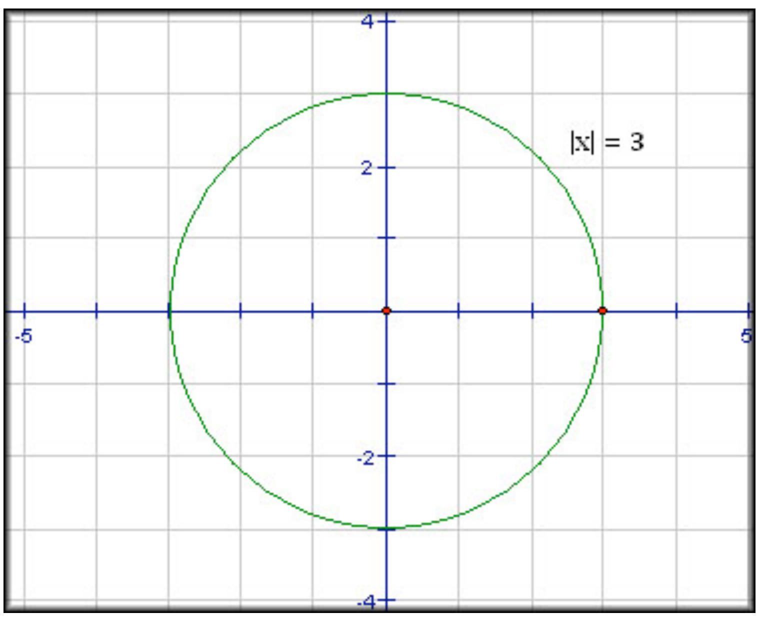

Figure 4

This idea can be extended to more interesting absolute value functions: $|3x + 1| = 5$. In the real numbers, there are two solutions, *x* = 4/3 and *x* = -2. In the complex numbers, however, we have infinite solutions. This can be seen in the following calculations:

If $x = a + bi$, then $3x + 1 = 3(a + bi) + 1 = 3a + 3bi + 1$. Grouping, so that $3a + 1$ is the real part and $3bi$ is the complex part, we get the following.

$$|3x+1| = |3a+1+3bi| = \sqrt{(3a+1)^2+(3b)^2} = \sqrt{9a^2+6a+1+9b^2}\,.$$

Returning to the original equation,

$$|3x+1| = 5 \rightarrow \sqrt{9a^2+6a+1+9b^2} = 5$$

$$9a^2+6a+1+9b^2 = 25$$

$$9a^2+6a+9b^2 = 24$$

$$a^2+\frac{2}{3}a+b^2 = \frac{8}{3}$$

$$a^2+\frac{2}{3}a+\frac{1}{9}+b^2 = \frac{8}{3}+\frac{1}{9}$$

$$\left(a+\frac{1}{3}\right)^2+b^2 = \left(\frac{5}{3}\right)^2$$

So, in the complex plane, solutions to linear absolute value equations are circles. In this case, we have a circle of radius 5/3 that is shifted 1/3 to the left on the real axis. There are infinite solutions to linear absolute value equations in the complex plane.

Mathematical Focus 4

A composite function with the same domain and co-domain may be composed of functions with different domains and co-domains.

The function represented by $f(x) = \left|\sqrt{x-5}\right|$ is a real function because it has domain and co-domain of $\Re$, the real numbers. Another way to express $f(x)$ is as the composition of three other functions, $r(x) = x-5$ which has domain and codomain of $\Re$, $s(x) = \sqrt{x}$ with domain $\Re$ and co-domain C, the complex numbers, and $t(x) = |x|$ which has domain C and co-domain $\Re$.

Since $f(x)$ is the composition of the functions r(x), s(x), and t(x) and can be expressed $f(x) = t \circ s \circ r(x) = t(s(r(x)))$. Thus, the value *f*(-10) from the prompt must the same as $t(s(r(-10)))$.

To compute the value of a composite function, we evaluate each function in turn, beginning with the innermost, $r(-10) = -10-5 = -15$. Next, we evaluate the function s(x) at x = -15: $s(x) = \sqrt{-15} = \sqrt{15}i$. Since the function t(x) has a co-domain in the real numbers, the final value of f(-10) will be a real number. The last step is to compute $t(\sqrt{15}i) = \left|\sqrt{15}i\right| = \sqrt{15} \approx 3.873$. The figure below (Figure 5) shows all the steps of the composition. Notice that the value we are computing starts as a real number, -10, but is sent into the complex numbers by function s(x), and then back to the (positive) real numbers by t(x).

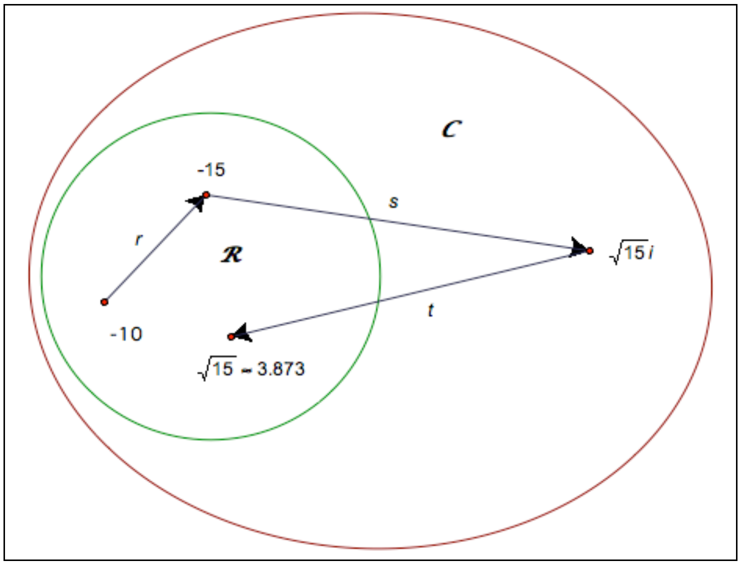

Figure 5

Post-Commentary

The 7th grade student whose conundrum is the focus of this prompt was originally asked to write a function with given characteristics. Given that he had probably not learned about complex numbers at this point, his confusion is justified. In the real number system, $f(x) = \left|\sqrt{x-5}\right|$ is undefined for values less than 5 and has a non-negative range. In the complex numbers, however, the function is *not* undefined for values less than 5. Because the student was using a calculator in complex mode, a CAS calculator in this case, he did not get the "undefined" result he expected. Instead, the calculator returned a real answer for an input value that the student expected to return an error message.

The issues of domain and co-domain raised in Focus 4 become especially salient when a CAS is employed. The technology allows for greater exploration by students but it also requires teachers and students to be more aware of the issues of domain. Teachers must also be more explicit about the number systems that are intended for use in solving tasks of this kind.